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# Voltage distribution in random systems

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*The problem of voltage and power distributions in random percolation systems is reviewed. Two-component percolation and the exponentially wide spectrum of conductances are analyzed. It is shown that voltage distribution in the two-component percolation has multipeak, multifractal structure. Distribution of powers* e *dissipated inside the percolation-like system with exponentially wide spectrum of resistances, i.e., the system in which bonds are occupied by conductances* $g \sim \exp(-\lambda x)$ *where* $\lambda \gg 1$ *and* x *is the random variable on [0, 1] is determined by numerical simulations in* $d = 3$ *dimensions. It is shown that distributions* $n(\ln e)$ *obtained for various values of system size* L *and parameter* $\lambda$ *collapse if displayed in coordinates* $\alpha = \ln[e(L/\xi)^2]$ *vs* $\ln[n(\ln e)/L^d]/\ln\xi + d$*, where* $\xi$ *is the percolation correlation length. The curve* $D(\alpha)$ *obtained by such a collapsing plays the role of the spectrum of fractal dimensions in the system with exponentially wide spectrum of conductances. © 1998 Elsevier Science Ltd. All rights reserved*

## 1. Introduction

Transport properties of heterogeneous media have recently attracted much interest. When the disorder of the medium is extremely large percolation theory[1] is a very efficient tool of investigation. The properties of electrical transport can be then described by the distribution of voltage drops in the so-called random resistor network (RRN). It turns out that various moments of this distribution have physical interpretations.[2] For example, the zero moment describes the mass of the percolating backbone, the second one describes the network conductance while the fourth is related to $1/f$ noise. It was shown that the distribution $n(\ln v^2)$ of the logarithm of voltage drops $v$ in RRN has a multifractal structure.[3] The term "multifractal" means that there is an infinite set of exponents $f(\alpha)$ which describe the power-law scaling, as a function of system size $L$, of different regions $\alpha \equiv \ln v^2/\ln L$ of the distribution, i.e. $n(\ln v^2)/\ln L \sim f(\alpha)$.

## 2. Two-component percolation

RRN may be considered as the limiting case of the more general two-component random resistor network (TCRRN) in which both components of the mixture take finite values of the conductance.[4] Namely, metallic conductance $g_m$ occupies bonds of $d$-dimensional lattice with probability $p$ whereas "insulating" conductance $g_d$ occupies bonds with probability $1-p$. Such TCRRN is a more realistic model of the metal-insulator composite in which non zero conductivity of insulator is taken into account. At the percolation threshold, i.e., for $p = p_c$ we have found that if the network of size $L$ is biased by unit voltage $V = 1$ then the distribution of the logarithm of voltage drops, $v$ that appear on lattice bonds is composed of several peaks shifted subsequently on $-\ln(v^2)$ axis by amount of $2\ln(hL^{1/(\nu\delta)}]$ where $h = g_d/g_m$, $\delta$ is the crossover exponent and $\nu$ is the percolation correlation length exponent.[5] This can be seen in Fig. 1 where the distribution of powers dissipated in TCRRN in which $g_m = 1$ and $g_d = h = 10^{-7}$ is shown. For powers $e$ dissipated in the network we have $e = g_m v^2 = v^2$ for metallic bonds and $e = g_d v^2 = hv^2$ for insulating bonds. This means that part $n_d$ which describes distribution of voltages dissipated in "insulating" bonds in power

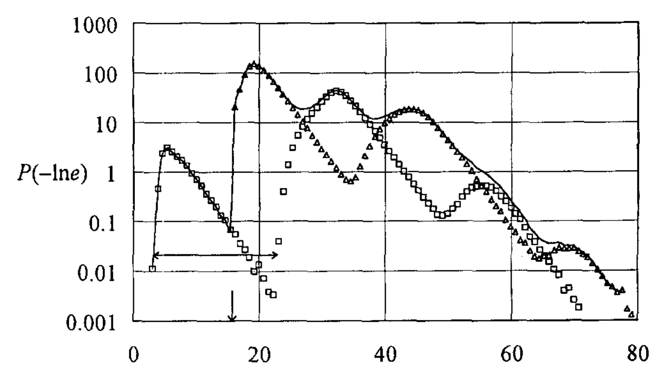


**Figure 1.** The distributions $P(-\ln e)$ of the logarithm of power dissipated in three-dimensional TCRRN of size $L = 8$ with $g_m = 1$ and $g_d = h = 10^{-7}$ (solid line). Points refer to powers dissipated in metallic bonds (□)—distribution $n_m(-\ln e)$, and insulating bonds (△)—distribution $n_d(-\ln e + \ln h)$. Solid line is the sum of the two. The arrow is placed at $-\ln e = -\ln h \cong 16$. The length of horizontal double arrow-headed line is $2\ln[hL^{1/(\nu\delta)}] \cong 20$ if the value of $1/(\nu\delta) = 3.07$ from Ref. [1] is used.

distribution is shifted by $\ln h$ towards lower powers and thus does not overlap that part $n_m$ which describes distribution of voltages/powers dissipated in metallic bonds. Like in the case of RRN the peaks that form the distribution in Fig. 1 have multifractal structure. This is seen for example in Fig. 2 where the data obtained for various values of system size $L$ are redrawn in coordinates $\ln n_m/\ln L$ vs $\alpha = -\ln(\upsilon^2)/\ln L$. After such rescaling they asymptotically take the shape of spectrum $f(\alpha)$. The shape of power distribution in Fig. 1 can be understood better in terms of qualitative analysis of transport processes which take place in the TCRRN. The first peak in the distribution $n_m$ is related to currents flowing in the backbone of the percolating cluster. If $g_d > 0$ currents start flowing in the insulating phase. The first peak in $n_d$ describes their distribution. This is however now the only effect. The other is that dangling ends and isolated metallic clusters, which in ideal ($g_d = 0$) RRN carry no currents, now carry currents that flow through the insulating phase. Thus they are of order $h$. This is the origin of the second peak in $n_m$: it describes the distribution of voltage drops in dangling ends, isolated clusters and all other metallic bonds which are "wetted" by currents when insulating phase takes finite value, $g_d > 0$. Similar qualitative explanation of further peaks both in $n_d$ and $n_m$ is also possible.

### 3. Exponentially wide spectrum of conductances

Until now we have considered two-component percolation. There is, however, also a very interesting case of continuous resistance distributions. Let us consider the case

$$g_i = g(x) = g_0 \exp(-\lambda x) \tag{1}$$

where $g_i$ is the conductance of the $i$-th bond and $x \in [0,1]$ is a random variable with a smooth distribution $N_x(x)$. An example which can be simplified to such problem is the high temperature hopping conduction. Although the system with exponentially wide spectrum of bond conductances defined by eqn (1) is not the classical percolation because the components do not exist—there is the well known approach[6] which allows one to reduce this problem to the two-component percolation. In this approach the network conductance $G$ is described by the so-called critical

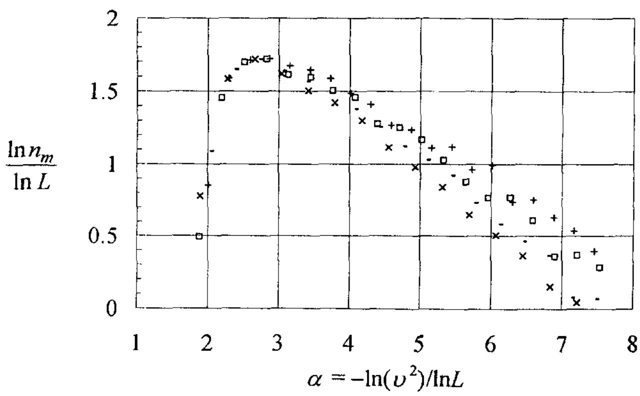


**Figure 2.** Multifractal spectrum of fractal dimensions obtained by rescaling power distributions (like those in Fig. 1) obtained for various values of network size: $L = 8$ (×), $L = 10$ (—), $L = 12$ (□), $L = 15$ (+). Only data which build up the first peak in $P$ (and/or $n_m$) in Fig. 1 were used. The value $\upsilon^2$ used on the horizontal axis is obtained as $\upsilon^2 = e/g_m = e$.

conductance $g_c$. The latter is defined as the smallest conductance which close up the percolating cluster, namely $g_c = g_0 \exp(-\lambda x_c)$, where

$$\int_0^{x_c} N_x(x)\,\mathrm{d}x = p_c$$

is the fraction of bonds required to form a cluster which starts spanning the network. In other words, the idea is to treat all the bonds with $g_i > g_c$ as "metallic phase" which form percolating cluster whereas bonds with $g_i < g_c$ as non conducting "insulating phase". A more detailed treatment leads one to the derivation of the scale $\xi$ over which the system have well-defined size independent conductivity. This scale is called correlation length and it was shown that

$$\xi \sim \lambda^{\nu}$$

where $\nu$ is the percolation correlation length exponent. Conductivity is then

$$\sigma = \xi^{2-d} g_c \sim \lambda^{-y} \exp(-\lambda x_c).$$

where

$$y = \nu(d-2).$$

For $d = 3$ numerical simulations of Tyč and Halperin[7] give $y = 0.6 \pm 0.1$, whereas our recent numerical estimate[8] is $y = 0.76 \pm 0.09$.

In order to determine voltage distribution in system with exponentially wide spectrum of conductances we have performed computer simulations of such system. In each computational step a simple cubic lattice ($d = 3$) of linear size $L$, in which bonds were occupied randomly in the way described by eqn (1) was generated. A uniform distribution $N_x(x) = 1$ was assumed for simplicity. Once the lattice was generated, conductances $g_i$ of all its bonds were stored in a band matrix of network equations and unit external voltage $V = 1$ was applied to the opposite walls of the lattice. Free boundary conditions were applied in the remaining two directions. Then the matrix was solved and voltages $\upsilon_i$ on all bonds of the lattice were determined. Eventually, powers $e_i = g_i \upsilon_i^2$ dissipated in the bonds were calculated and their populations were gathered into bins of the width of $\Delta \ln e = 0.77$. We have performed simulations for various values of parameter $\lambda = 40, 50, 60, 70$ and for various values of the lattice size $L = 8$, 10, 11. For a pair of these parameters fixed, several thousands of network realizations were generated and distributions $n(\ln e)$ were averaged. Results are shown in Fig. 3. Let us recall that the multipeak, multifractal structure of voltage distribution was found for a TCRRN at the percolation threshold. In this case the percolation correlation length is infinite and the system is always in the fractal regime. In the present case of the system with exponentially wide spectrum of conductances the correlation length is always finite, $\xi \sim \lambda^{\nu}$. This influences the shape of calculated distribution of power. For example, in the limit $\lambda \to \infty$ we have also $\xi \to \infty$ and for finite $L$ we expect the distribution gets the shape of multifractal type. On the contrary for finite $\xi$ and in the limit $L \to \infty$ the system becomes macroscopically homogeneous and we expect that internal voltages form the distribution which is peaked at the value of voltage of $V_{\xi} = V\xi/L$ and scales with system size in Euclidean way. Namely, $n(\ln \upsilon^2) \sim (L/\xi)^d$. Such scaling is obvious when we partition the

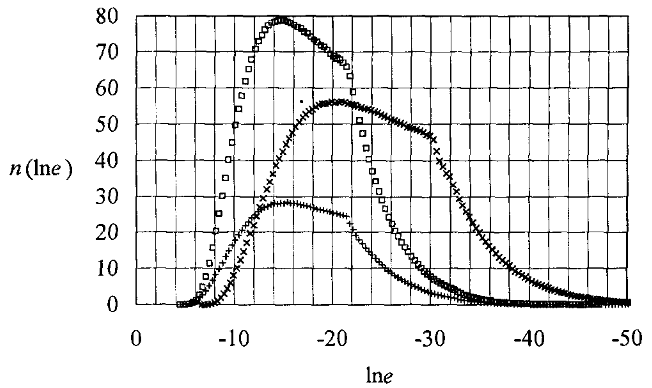


**Figure 3.** Power distributions in systems with exponentially wide spectrum of conductances. Symbols refer to the results of numerical simulations performed for various values of system size $L$ and parameter $\lambda$: $L = 11$, $\lambda = 50$ (□), $L = 11$, $\lambda = 70$ (×), $L = 8$, $\lambda = 50$ (+).

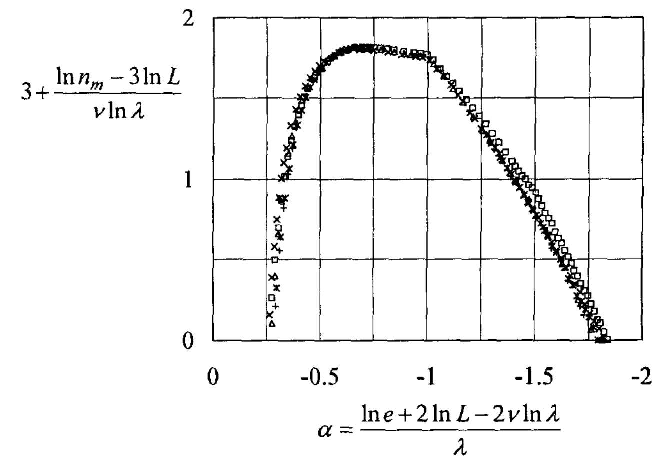


**Figure 4.** Spectrum $D(\alpha)$ of fractal dimensions that characterize power distribution in the systems with exponentially wide spectrum of conductances. Symbols refer to the systems of various values of size $L$ and parameter $\lambda$: $L = 8$, $\lambda = 50$ (□), $L = 11$, $\lambda = 70$ (×), $L = 11$, $\lambda = 50$ (△), $L = 11$, $\lambda = 40$ (+) and $L = 10$, $\lambda = 40$ (*). In the rescaling of the data on the axes the value $\nu = 0.76$ from Ref. [8] was used.

system of size $L$ biased by a unit voltage $V = 1$ V into hypercubes of size $\xi$. Each hypercube is then biased by voltage $V_\xi$. The total number, $n(\ln e)$, of bonds in which the power $e$ is dissipated is then

$$n(\ln e) = \left(\frac{L}{\xi}\right)^d n_\xi(\ln e),$$

where $n_\xi(\ln e)$ is the number of bonds in which the power $e$ is dissipated inside the hypercube of size $\xi$. Let us now note that the voltage $V_\xi$ which biases the hypercube of size $\xi$ appears on the bond with critical conductance $g_c$. In this bond the maximum power of $e_{\max} \sim V_\xi^2 \exp(-\lambda x_c)$ is dissipated. Thus, if we want the distributions obtained for various $L$ and $\lambda$ collapse after a rescaling we must use the quantity $\alpha = \ln[e(L/\xi)^2]/\lambda$ on the horizontal axis. In this case the high-power part of the distributions should approach the value of $\alpha_{\max} = -x_c$. Let us further assume that, like in the case of TCRRN, the number of bonds in which the power $e$ is dissipated contained within a hypercube of size $\xi$ scale with its size in a power-law manner $n_\xi(\ln e) = \xi^{D(\alpha)}$, where $D(\alpha)$ is the $\alpha$-dependent fractal dimension. Consequently $n(\ln e)$ scale with $L$ and $\xi$ as

$$n(\ln e) \sim \left(\frac{L}{\xi}\right)^d \xi^{D(\alpha)} = L^d \xi^{D(\alpha)-d}$$

and this means the proper quantity we should use on vertical axis to rescale the data is $\ln[n(\ln e)/L^d]/\ln\xi + d$. In this case the distributions obtained for different $L$ and $\lambda$ should collapse giving the single curve $D(\alpha)$ which should be regarded as the spectrum of fractal dimensions characterizing distribution of powers dissipated in the system with exponentially wide spectrum of conductances. Figure 4 confirm our analysis. We may observe collapsing of the distributions obtained for various $L$ and $\lambda$. As we expect they approach the value of $\alpha_{\max} = -0.25$ in excellent agreement with the value of percolation threshold in simple cubic lattice, $x_c = 0.25$.

## 4. Summary

The problem of voltage and power distributions in random percolation systems is reviewed. Distribution of internal powers dissipated in the percolation-like system with exponentially wide spectrum of conductances was determined by numerical simulations in $d = 3$ dimensions. It was shown that distributions obtained for various values of system size $L$ and parameter $\lambda$ collapse if displayed in coordinates $\alpha = \ln[e(L/\xi)^2]$ vs $\ln[n(\ln e)/L^d]/\ln\xi + d$. The curve $D(\alpha)$ obtained by such a collapsing plays the role of the spectrum of fractal dimensions in the system with exponentially wide spectrum of conductances.

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